



TRAVELING WAVE DYNAMICS IN A TRANSLATING STRING COUPLED TO STATIONARY CONSTRAINTS: ENERGY TRANSFER AND MODE LOCALIZATION

S.-Y. LEE* AND C. D. MOTE, JR.

*Dynamics Stability Laboratory, Department of Mechanical Engineering,
University of California at Berkeley, Berkeley, CA 94720, U.S.A.*

(Received 19 August 1996, and in final form 2 September 1997)

Energy transfer and mode localization in a translating string, coupled to a stationary spring-mass-damper system, are analyzed using traveling waves. The string tension and the non-conservative centrifugal force at the constraint lead to energy transfer between the translating continuum and the stationary constraint. By calculating energy contained in a harmonic wave before and after interacting with the constraint, energy transferred at the constraint is quantified in terms of constraint parameters. At an undamped constraint, energy is transmitted when a downstream (forward) wave impinges on the constraint. At a damped constraint, energy dissipated by damping, as well as energy flux by the tension and centrifugal force, contribute to the energy variation at the coupling point. When the damping coefficient of the damped constraint equals twice the impedance of the string, vibration energy is maximally dissipated. Asymmetry caused by the constraint localizes vibration modes into a downstream or upstream region. The degree of localization in each vibration mode is described in terms of the reflection coefficient and the constraint location. The effect of the constraint parameters on the mode localization and veering of natural frequency loci are also examined.

© 1998 Academic Press Limited

1. INTRODUCTION

A class of flexible translating mechanical systems used for transmitting power, material or information is referred to as *axially moving materials*. Such systems include belts, chains, magnetic tapes, paper sheets and threads, composite and textile fibers, pipes transporting fluids, flexible robotic manipulators with prismatic joints, and flexible appendages on spacecraft. The extensive research in the axially moving systems is summarized in review articles by Wickert and Mote [1], Wang and Liu [2] and Paidoussis and Li [3].

The translating string in contact with a stationary constraint is a common model for a magnetic tape passing across a recording head or a band saw translating over a fixed guide bearing. The linear free and forced responses of a translating string over an elastic foundation were studied by eigenvalue analyses [4] and the transfer function method [5]. It was shown that the elastic foundation does not alter the critical speed in the linear model. Cheng and Perkins [6] examined the linear vibration and stability of a string translating through an elastically supported, dry friction guide. The transient response of a translating string with an attached inertia [7] and a traveling damped linear oscillator [8] were studied using a Green's function formulation. Zhu and Mote [9] have studied transient response

* Now at the Department of Mechanical Engineering, Sogang University, Seoul, Korea.

of a translating string constrained to a fixed linear oscillator and derived the exact expression for the constraint force in the constrained translating string. Recently, Chen [10] also examined the natural frequencies and stability of a traveling string in contact with a stationary load system.

An analogous problem with a similar governing equation is the vibration of a constrained rotating circular string. The idealized rotating string is a simplified model of a computer memory disk, a guided circular saw or a turbine disk. These non-dispersive models provide some insight to the dynamics of a translating or rotating, flexible medium when subjected to fixed constraint forces. The free response of a rotating circular string coupled to various stationary constraints was examined by Schajer [11] for a point elastic restraint, Xiong and Hutton [12] for the distributed elastic constraint, and Yang and Hutton [13] for various stationary constraints.

One of the interesting features observed in a translating string is that the free vibration energy shows a periodic variation [14, 15]. For a non-translating undamped string, the total energy is constant. However under translation, energy transfers into or out of the moving continuum at a boundary support. Wickert and Mote [16] showed that energy flux occurring at a fixed boundary of the traveling string is the product of the string tension and the convective component of the velocity. Energy transferred between the translating string and various boundary supports, was quantified using the traveling wave method [17].

Eigenvalues (natural frequencies) are often plotted versus a system parameter creating a family of frequency loci. The effect of the system parameter on the corresponding vibration modes has been studied by many researchers for nearly periodic structures [18, 19] and gyroscopic systems [6, 20]. Relatively small structural irregularities may result in large changes in dynamic properties of a structure and localize the associated vibration modes. For a translating string, a stationary constraint attached to the moving continuum may cause asymmetry of the system and localize vibration modes into a downstream or upstream region.

First, energy transfer observed in the translating string coupled to a stationary constraint of a spring-mass-damper system is examined in this paper. Energy transferred at the constraint is represented in terms of the constraint parameters and impedance of the translating string. Second, the degree of localization of each vibration mode is derived in terms of the reflection coefficient and the location of the constraint. Veering of frequency loci associated with mode localization are also quantified as a function of the magnitude of disorder and the coupling factor.

2. SYSTEM MODEL

Consider a uniform string translating at constant speed and tension between two supports separated by a distance L . The string passes through a frictionless, fixed constraint modelled by stiffness, inertia and viscous damping at $X = X_c$, as shown in

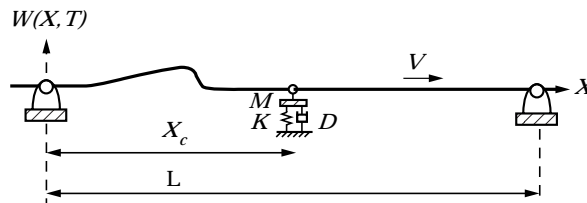


Figure 1. Schematic of a translating string coupled to a stationary constraint of a spring-mass-damper system.

Figure 1. Under the assumptions that the transverse motion is small compared to the length L and the initial tension in the string is sufficiently large that its vibration during small amplitude vibration is negligible, the linear equation of transverse motion $W(X, T)$ of the traveling string becomes

$$\rho \left(\frac{\partial^2 W}{\partial T^2} + 2V \frac{\partial^2 W}{\partial X \partial T} + V^2 \frac{\partial^2 W}{\partial X^2} \right) - P \frac{\partial^2 W}{\partial X^2} = - \left(M \frac{\partial^2 W}{\partial T^2} + D \frac{\partial W}{\partial T} + KW \right) \delta(X - X_c), \quad (1)$$

where $0 < X < L$, ρ is the linear density of the string, P is the constant tension, K is the spring stiffness, D is the linear damping coefficient and M is the mass of the constraint. Introduction of the dimensionless variables

$$\begin{aligned} x &= \frac{X}{L}, & w &= \frac{W}{L}, & t &= \frac{T}{L} \left(\frac{P}{\rho} \right)^{1/2}, & v &= V \left(\frac{\rho}{P} \right)^{1/2}, \\ x_c &= \frac{X_c}{L}, & m &= \frac{M}{\rho L}, & d &= \frac{D}{(P\rho)^{1/2}}, & k &= K \frac{L}{P} \end{aligned} \quad (2)$$

into equation (1) yields the normalized equation of motion

$$w_{tt} + 2vw_{xt} - (1 - v^2)w_{xx} = -(mw_{tt} + dw_t + kw)\delta(x - x_c), \quad 0 < x < 1, \quad (3)$$

where each subscript denotes partial differentiation. The inlet and outlet boundaries are fixed:

$$w(0, t) = w(1, t) = 0. \quad (4)$$

At $x = x_c$, continuity and force balance conditions require

$$\begin{aligned} w(x_c^-, t) - w(x_c^+, t) &= 0, \\ (1 - v^2)\{w_x(x_c^-, t) - w_x(x_c^+, t)\} + kw(x_c, t) + dw_t(x_c, t) + mw_{tt}(x_c, t) &= 0. \end{aligned} \quad (5)$$

The critical transport speed for divergence instability, $v_c = 1$, is determined from the time-independent terms in the linear model (3). In this paper, v remains constant and subcritical. Applying the eigenvalue problem of the form, $w(x, t) = W(x) e^{i\omega t}$, into equation (3) yields the characteristic equation

$$(m\lambda^2 + d\lambda + k) \sinh \frac{x_c \lambda}{1 - v^2} \sinh \frac{(1 - x_c) \lambda}{1 - v^2} + \lambda \sinh \frac{\lambda}{1 - v^2} = 0. \quad (6)$$

Roots of equation (6) are generally complex values, $\lambda = v + i\omega$. Here ω is real and denotes the dimensionless natural frequency of the system. At a non-dissipative constraint ($d = 0$), the eigenvalue becomes purely imaginary ($v = 0$). For a string without the constraint ($m = d = k = 0$), the solution of equation (6) recovers the natural frequencies of the classical moving threadline, $\omega_n = n\pi(1 - v^2)$ where $n = 1, 2, 3, \dots$ [21, 22]. For a limiting case of the spring stiffness, $k \rightarrow \infty$, the upstream and downstream string segments decouple, and the natural frequencies become $\omega_m = m\pi(1 - v^2)/x_c$ and $\omega_n = n\pi(1 - v^2)/(1 - x_c)$ where $m, n = 1, 2, 3, \dots$. These are the natural frequencies of the two moving threadlines of lengths x_c and $(1 - x_c)$.

3. ENERGY EXPRESSION BY TRAVELING WAVES

In general, the free vibration response of the translating string is described through modal superposition of normal modes (standing waves). However, the modal approach

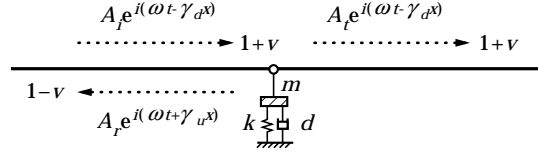


Figure 2. Wave reflection and transmission for downstream incident waves in a translating continuum–stationary constraint system.

becomes inefficient to analyze energy transfer between the moving continuum and the constraint. In this study, traveling waves rather than standing waves are used to describe the energy transfer.

3.1. TRAVELING WAVE CHARACTERISTICS

A wave propagating along one-dimensional continua has the form

$$w(x, t) = A e^{i(\omega t - \gamma x)}, \quad (7)$$

where ω and γ are frequency and wavenumber of the wave. The dimensionless frequency is related to the dimensional frequency Ω by $\omega = \Omega L \sqrt{(\rho/P)}$. For an infinite translating string, the transverse harmonic motion is represented by two independent traveling waves:

$$w(x, t) = A_d e^{i(\omega t - \gamma_d x)} + A_u e^{i(\omega t + \gamma_u x)}. \quad (8)$$

Here γ_d and γ_u are wavenumbers of downstream (forward) and upstream (backward) traveling waves. The dispersion relation obtained from equations (3) and (7),

$$(1 - v^2)\gamma^2 + 2v\gamma k - \omega^2 = 0, \quad (9)$$

leads to the wavenumbers of downstream and upstream waves:

$$\gamma_d = \frac{\omega}{1 + v}, \quad \gamma_u = \frac{\omega}{1 - v}. \quad (10)$$

The phase velocities of the waves are, respectively, $c_d = 1 + v$ and $c_u = 1 - v$. When an incident wave of amplitude A_i , traveling downstream, meets the constraint, waves reflected from and passing through the constraint are described by $A_r e^{i(\omega t + \gamma_u x)}$ and $A_t e^{i(\omega t - \gamma_d x)}$, as shown in Figure 2. The subscripts i , r and t identify the incident, reflected and transmitted waves. Wave scattering at the discontinuity is usually characterized by the reflection and transmission coefficients, $r = A_r/A_i$ and $t = A_t/A_i$. When the discontinuity has no external energy source or sink, energy conservation requires [23]

$$rr^* + tt^* = 1, \quad r^*t + rt^* = 0, \quad (11)$$

and the coefficients take the forms

$$r = |r| e^{i\phi_r}, \quad t = |t| e^{i(\phi_r + \pi/2)}, \quad (12)$$

where the asterisk denotes complex conjugation.

3.2. ENERGY TRANSFER MECHANISM

For a non-translating string coupled to an undamped constraint ($d = 0$), free vibration energy is always constant. If the string translates, then energy flux between the moving continuum and the constraint occurs and the vibrating energy varies with time. The constraint forces, causing the energy variation, can be identified by considering traveling

waves. For a downstream traveling wave, $w(x, t) = A_d e^{i(\omega t - \gamma_d x)}$, the transverse forces applying on a constraint at $x = 0$ are

$$F_m = -mw_t(0, t) = m\omega^2 A_d e^{i\omega t}, \quad F_c = -dw_t(0, t) = -i d\omega A_d e^{i\omega t}, \quad (13, 14)$$

$$F_k = -kw(0, t) = -kA_d e^{i\omega t}, \quad F_n = -(1 - v^2)w_x(0, t) = i(1 - v^2)\gamma_d A_d e^{i\omega t}. \quad (15, 16)$$

Here F_n is the transverse component of the string tension plus the centrifugal force. The centrifugal force is produced by the continuum passing through the constraint. In the string model, there is no continuity for the slope at the constraint point unlike a beam model. Thus, the slope in equation (16) is the average one at the point: $w_x(0, t) = \{w_x(0^+, t) + w_x(0^-, t)\}/2$. Since an element of the translating continuum located at position x changes with time, the instantaneous transverse velocity of the particle is $w_t + vw_x$ in inertial co-ordinates. For the downstream wave, the instantaneous velocity at the constraint becomes

$$v_p(t) = w_t(0, t) + vw_x(0, t) = i(\omega - v\gamma_d)A_d e^{i\omega t} = \frac{\omega}{1 + v} iA_d e^{i\omega t}. \quad (17)$$

Since F_m and F_k are out of phase by $\pi/2$ with the transverse velocity v_p , neither a spring nor mass component causes energy flux at the constraint over a cycle. However, the damping force F_c is always out of phase by π with v_p and results in energy dissipation at the constraint. Equations (16) and (17) show that the tension and centrifugal force F_n , is in phase with the material velocity v_p at the subcritical speed range. Energy is then transmitted into the system when the forward wave impinges on the constraint (Figure 3(a)). In this case, translation energy of the moving string, produced from the driving system, is added to the vibrating energy. Therefore, even if the constraint is fixed, the force F_n leads to energy transfer because the material transverse velocity at the constraint, $v_p(t) = vw_x(0, t)$, is not always zero [17]. For a backward traveling wave, $w(x, t) = A_u e^{i(\omega t + \gamma_u x)}$, the transverse force

$$F_n = -i(1 - v^2)\gamma_u A_u e^{i\omega t} \quad (18)$$

is out of phase by π with the transverse velocity

$$v_p(t) = w_t(0, t) + vw_x(0, t) = i(\omega + v\gamma_u)A_u e^{i\omega t}. \quad (19)$$

The energy transfer mechanism is contrary to the case of the forward wave. Energy is always dissipated when the upstream wave interacts with the constraint (Figure 3(b)).

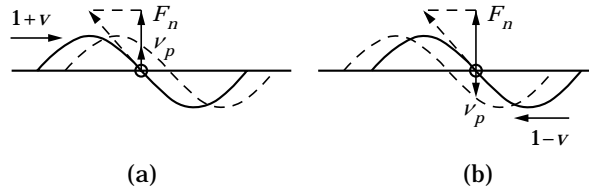


Figure 3. The effect of the force F_n on energy flux at the constraint: (a) forward incident wave, (b) backward wave.

3.3. ENERGY COEFFICIENTS

In a non-translating string, rr^* and tt^* are generally used to describe energy ratios of the reflected and transmitted waves to the incident wave. However, the coefficients are inappropriate to describe energy ratios in the translating continuum, because translation leads to different impedances (wavenumbers) between downstream and upstream waves. In this section, energy coefficients are defined to quantify the energy transfer produced at the constraint of the translating string. The total energy per unit length of the translating string is the sum of the kinetic and potential energy densities:

$$\hat{E} = \frac{1}{2}\rho(w_t + vw_x)^2 + \frac{1}{2}Pw_x^2. \quad (20)$$

The energy contained in one wavelength of a traveling wave $A e^{i(\omega t - \gamma x)}$ is

$$E_\lambda = \int_x^{x+\lambda} \hat{E} dx = \pi\omega \mathcal{L} A A^*, \quad (21)$$

where $\mathcal{L} = P/c$ is the mechanical impedance of the string. When a wave with impedance \mathcal{L}_i is incident on the constraint, part of the wave will be reflected into a region with different impedance \mathcal{L}_r , and the remaining wave transmits through the constraint into a region with the same impedance ($\mathcal{L}_t = \mathcal{L}_i$). Energy, transferred into the wave at the constraint over a cycle becomes

$$\Delta E = E_r + E_t - E_i = \pi\omega(\mathcal{L}_r A_r A_r^* + \mathcal{L}_t A_t A_t^* - \mathcal{L}_i A_i A_i^*). \quad (22)$$

The energy reflection coefficient R is then defined as the fraction of the incident wave energy that is reflected [17]:

$$R = \frac{E_r}{E_i} = \frac{\mathcal{L}_r A_r A_r^*}{\mathcal{L}_i A_i A_i^*} = \frac{\gamma_r}{\gamma_i} rr^*. \quad (23)$$

The energy transmission coefficient T is also defined as

$$T = \frac{E_t}{E_i} = \frac{\mathcal{L}_t A_t A_t^*}{\mathcal{L}_i A_i A_i^*} = tt^*. \quad (24)$$

The total energy coefficient,

$$E = \frac{E_r + E_t}{E_i} = \frac{\gamma_r}{\gamma_i} rr^* + tt^*, \quad (25)$$

gives the energy ratio of the transmitted plus reflected waves relative to the incident wave.

3.3.1. Undamped constraint

For the case of $d = 0$, only F_n causes energy transfer at the constraint. From the energy coefficient (25), work done by the force is calculated and its ratio to the incident wave energy is determined by the energy flux coefficient

$$\mathcal{F} = \frac{\Delta E}{E_i} = \frac{\mathcal{L}_r - \mathcal{L}_i}{\mathcal{L}_i} \left(\frac{A_r}{A_i} \right) \left(\frac{A_r}{A_i} \right)^* = \frac{\gamma_r - \gamma_i}{\gamma_i} rr^*. \quad (26)$$

As predicted in section 3.2, the coefficient becomes positive and negative for the forward and backward incident waves,

$$\mathcal{F}_f = \frac{2v}{1-v} rr^* \geq 0, \quad \mathcal{F}_b = -\frac{2v}{1+v} rr^* \leq 0, \quad (27)$$

respectively. Note that the energy transfer is proportional to the wave reflected from the constraint. Using the flux coefficient, the total energy coefficient (25) is rewritten as $E = 1 + \mathcal{F}$.

3.3.2. Damped constraint

For the case of $d \neq 0$, an additional energy transfer done by the force F_c in equation (14) must be considered. The absorption coefficient, defined as

$$\alpha = 1 - rr^* - tt^*, \quad (28)$$

determines energy dissipated by damping relative to the incident wave energy. The coefficient satisfies $0 \leq \alpha \leq 1$. Without damping, $\alpha = 0$. In this case, the energy flux coefficient becomes

$$\mathcal{F} = \frac{\Delta E}{E_i} - \alpha = \frac{\gamma_r - \gamma_i}{\gamma_i} rr^*, \quad (29)$$

and it is identical to the undamped case (26). The total energy coefficient (25) is also represented by $E = 1 - \alpha + \mathcal{F}$.

4. ENERGY TRANSFER AT THE CONSTRAINT

The energy coefficients, derived in the previous section, require the values of r and t to calculate energy transferred at the constraint. The values are determined from the boundary conditions of the constraint. When a downstream traveling wave is incident on the constraint, the wave response is represented by

$$w(x, t) = \begin{cases} A_i e^{i(\omega t - \gamma_d x)} + A_r e^{i(\omega t + \gamma_u x)} & : x \leq 0^- \\ A_t e^{i(\omega t - \gamma_d x)} & : x \geq 0^+ \end{cases}. \quad (30)$$

Substitution of equation (30) into the geometric and force balances at the constraint,

$$w(0^-, t) = w(0^+, t),$$

$$(1 - v^2)\{w_x(0^-, t) - w_x(0^+, t)\} + mw_u(0, t) + dw_i(0, t) + kw(0, t) = 0, \quad (31)$$

gives the complex reflection and transmission coefficients in terms of the constraint parameters such as

$$r = \frac{A_r}{A_i} = -\frac{(k - m\omega^2) + id\omega}{(k - m\omega^2) + i(d + 2)\omega}, \quad t = \frac{A_t}{A_i} = \frac{i2\omega}{(k - m\omega^2) + i(d + 2)\omega}. \quad (32)$$

These ratios are independent of speed v and wavenumbers γ_d and γ_u , because an observer moving with the string measures the same amplitude and phase as a fixed observer. Therefore, the reflection and transmission coefficients for an upstream wave or a non-translating string are equal to those in equation (32).

4.1. SPRING CONSTRAINT

For the constraint of a spring only ($m = d = 0$), the reflection and transmission coefficients at the constraint are, from equation (32),

$$r = -\frac{k}{k + i2\omega}, \quad t = \frac{i2\omega}{k + i2\omega}. \quad (33)$$

The elastic constraint is transparent to high frequency. The corresponding energy reflection coefficients for forward and backward incident waves are

$$R_f = \left(\frac{1+v}{1-v} \right) \frac{k^2}{k^2 + 4\omega^2}, \quad R_b = \left(\frac{1-v}{1+v} \right) \frac{k^2}{k^2 + 4\omega^2}. \quad (34)$$

For the forward wave, energy is transferred into the system and its magnitude relative to the incident energy is represented in terms of system parameters using equation (26):

$$\mathcal{F}_f = \frac{2v}{1-v} \left(\frac{k^2}{k^2 + 4\omega^2} \right). \quad (35)$$

For the backward wave, the wave energy decreases at the constraint because of $\mathcal{F}_b = -\mathcal{F}_f < 0$. For $k \rightarrow \infty$, the waves are completely reflected with a phase change of π ($r = -1$ and $t = 0$). In the limiting case, the energy coefficient for the forward wave becomes maximum, $E_f = (1+v)/(1-v)$. For the backward wave, it is minimum, $E_b = (1-v)/(1+v)$.

4.2. MASS CONSTRAINT

When the constraint is modelled by a point mass ($k = d = 0$), the reflection and transmission coefficients are

$$r = \frac{m\omega}{-m\omega + i2}, \quad t = \frac{i2}{-m\omega + i2}. \quad (36)$$

Wave propagation by the point mass is complementary to the elastic constraint. The constraint reflects harmonic waves of high frequency and transmits low frequency ones. For forward and backward incident waves, the resultant energy flux ratios, \mathcal{F}_f and \mathcal{F}_b , are shown in Table 1. The energy reflection and transmission coefficients are plotted in Figures 4(a) and (b) when the constraint has a spring of $k = 10$ or a point mass of $m = 0.1$. Here translation speed is $v = 0.5$. High frequency waves transmit through the elastic constraint and R_f and R_b increase with frequency. The frequency, at which one half of the incident wave at the spring constraint is transmitted ($T = 0.5$), is $\omega = k/2$. For the point mass, the frequency of $T = 0.5$ is $\omega = 2/m$. The total energy coefficients under the constraint of the spring or point mass are shown in Figures 4(c) and (d). The total energy coefficients satisfy $1 \leq E_f \leq 3$ and $0.5 \leq E_b \leq 1$.

TABLE 1
Energy bounds and energy flux coefficients of the translating string

Constraint	Forward incident wave		Backward incident wave	
Spring	$1 \leq E_f \leq \frac{1+v}{1-v}$	$\mathcal{F}_f = \frac{2vk^2}{(1-v)(k^2 + 4\omega^2)}$	$\frac{1-v}{1+v} \leq E_b \leq 1$	$\mathcal{F}_b = -\frac{2vk^2}{(1+v)(k^2 + 4\omega^2)}$
Mass	$1 \leq E_f \leq \frac{1+v}{1-v}$	$\mathcal{F}_f = \frac{2vm^2\omega^2}{(1-v)(m^2\omega^2 + 4)}$	$\frac{1-v}{1+v} \leq E_b \leq 1$	$\mathcal{F}_b = -\frac{2vm^2\omega^2}{(1+v)(m^2\omega^2 + 4)}$
Damper	$\frac{1+v}{2} \leq E_f \leq \frac{1+v}{1-v}$	$\mathcal{F}_f = \frac{2vd^2}{(1-v)(d+2)^2}$	$\frac{1-v}{2} \leq E_b \leq 1$	$\mathcal{F}_b = -\frac{2vd^2}{(1+v)(d+2)^2}$

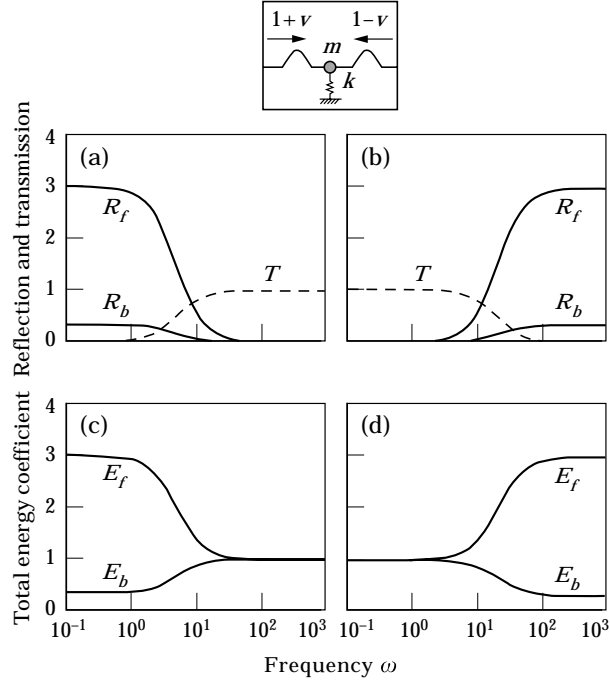


Figure 4. The effect of frequency ω on energy coefficients at a spring and a mass constraint when $v = 0.5$: the reflection and transmission coefficient: (a) $m = 0$, $k = 10$, (b) $m = 0.1$, $k = 0$; the total energy coefficient: (c) $m = 0$, $k = 10$, (d) $m = 0.1$, $k = 0$.

4.3. SPRING-MASS CONSTRAINT

Consider the case that the constraint is a spring-mass oscillator ($d = 0$). At the constraint,

$$r = -\frac{k - m\omega^2}{(k - m\omega^2) + i2\omega}, \quad t = \frac{i2\omega}{(k - m\omega^2) + i2\omega}. \quad (37)$$

The energy flux coefficients and the bounds of the coefficient E are similarly represented in terms of the constraint parameters. The energy coefficients of the spring-mass constraint with $k = 10$ and $m = 0.1$ are shown in Figures 5(a)–(c). At small ω , the incident wave largely reflects from the constraint. When the frequency equals the natural frequency of the oscillator ($\omega = \sqrt{k/m} = 10$), the incident wave transmits through the oscillator without reflection ($r = 0$, $t = 1$). At the frequency, the total wave energy remains the same and the energy flux coefficient vanishes. As ω increases further, the reflected wave increases again.

4.4. DAMPED CONSTRAINT

4.4.1. Energy transfer

For a viscously damped constraint ($k = m = 0$), the reflection and transmission coefficients at the coupled point become

$$r = -\frac{d}{d+2}, \quad t = \frac{2}{d+2}. \quad (38)$$

The coefficients are real and independent of frequency. It is also satisfied that $r^2 + t^2 < 1$. When a forward harmonic wave interacts with the constraint, the associated energy coefficients,

$$R_f = \frac{1+v}{1-v} \left(\frac{d}{d+2} \right)^2, \quad T = \left(\frac{2}{d+2} \right)^2, \quad (39)$$

give the energy ratios of the reflected and transmitted waves to the incoming wave. R_f approaches $(1+v)/(1-v)$ and T decreases, as d increases. Two types of forces are associated with energy transfer at the damped constraint. The ratio of energy dissipated by damping to the incident energy is represented by

$$\alpha = 1 - rr^* - tt^* = \frac{4d}{(d+2)^2}. \quad (40)$$

The string tension and centrifugal force are responsible for the second energy flux at the

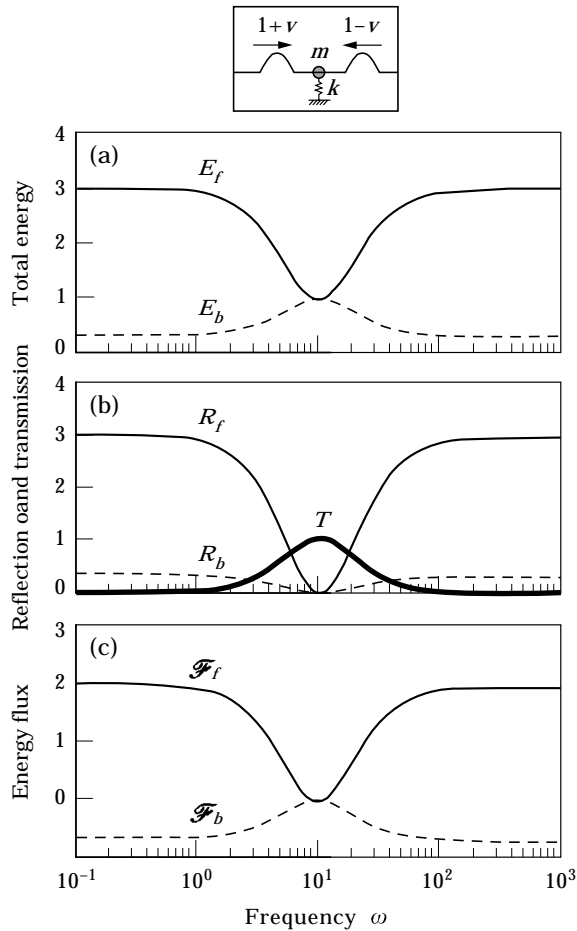


Figure 5. Energy coefficients at a spring-mass constraint ($k = 10$ and $m = 0.1$): (a) total energy coefficients, (b) reflection and transmission coefficients, (c) energy flux coefficients.

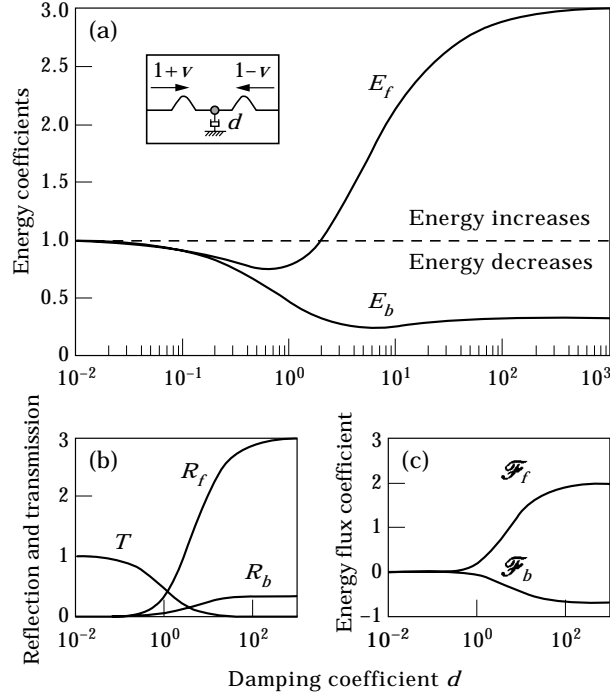


Figure 6. Energy coefficients at a damped constraint when $v = 0.5$: (a) total energy coefficients, (b) energy flux coefficients, (c) reflection and transmission coefficients.

constraint, as shown in section 3.2. The energy transfer divided by the incident energy is quantified by the energy flux coefficient

$$\mathcal{F}_f = \frac{2vd^2}{(1-v)(d+2)^2}. \quad (41)$$

For the forward wave, the two energy fluxes have opposite signs. The critical value of the damping coefficient, where the total energy transfer becomes zero ($\mathcal{F}_f = \alpha$), is

$$d_c = \frac{2(1-v)}{v}. \quad (42)$$

It is noted that, for $d > d_c$, the incident wave energy increases after interacting with the damped constraint ($E_f > 1$). The destabilizing effect of dissipation is analogous to a similar phenomenon observed in pipes conveying fluid [3, 24].

For a backward traveling wave, the energy flux coefficient $\mathcal{F}_b = -\mathcal{F}_f$ is always negative. Energy flux by the tension and centrifugal force is opposite to the case of the forward wave. Both energy fluxes (\mathcal{F}_b and α) contribute to energy decrease. Therefore, the total energy coefficient, $E_b = 1 - \alpha + \mathcal{F}_b$, is always less than unity. The energy bounds and the energy flux coefficients for the various constraint conditions are summarized in Table 1.

4.4.2. Optimal damping

Two optimal values of damping are considered here. One is determined by minimizing the total wave energy (E). The optimal damping coefficients, minimizing the energy

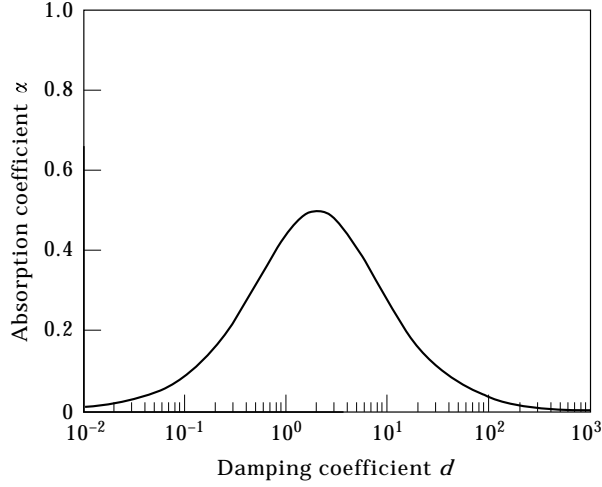


Figure 7. The absorption coefficient α at a damping support when $v = 0.5$. The optimal damping coefficient for maximal energy dissipation is $d = 2$.

coefficients E_f and E_b at the damped constraint, become

$$d_f = \frac{2(1-v)}{1+v}, \quad d_b = \frac{2(1+v)}{1-v}. \quad (43)$$

With these values, $E_f = (1+v)/2$ and $E_b = (1-v)/2$. The bounds of the energy coefficients are in Table 1.

The second optimal damping coefficient is obtained by maximizing energy dissipation by the damping element. When $d_{opt} = 2$, the energy absorption coefficient has the largest value ($\alpha = 0.5$). It is concluded that energy dissipated by damping cannot exceed 50% of the incoming wave energy, because both reflection and transmission coefficient cannot vanish simultaneously ($0.5 \leq rr^* + tt^* \leq 1$). Since the dimensional value of d_{opt} is $D = (P\rho)^{1/2}d = 2\mathcal{Z}$, the optimal value equals twice the impedance of the string at $v = 0$. For the case of damping attached at a boundary of the translating string, the optimal value of damping equals the impedance of the string. Further, the optimal boundary damping can dissipate all the incident energy [25].

The energy coefficients at the damped constraint are plotted in Figure 6 when $v = 0.5$. The total energy of a forward incident wave increases ($E_f > 1$) for $d > d_c = 2$, where \mathcal{F}_f exceeds α and E_f is minimized at $d = d_f = 2/3$. For a backward wave, \mathcal{F}_b is negative and E_b is minimum at $d = d_b = 6$. The energy coefficients are within the bounds: $1 \leq E_f \leq 3$ and $0.5 \leq E_b \leq 1$. Both R_f and \mathcal{F}_f increase as d increases. The associated absorption coefficients

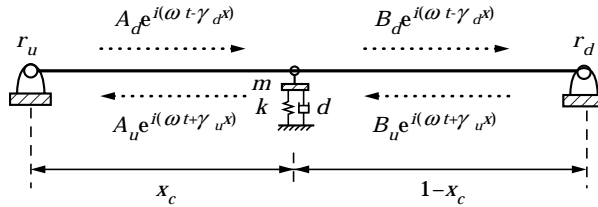


Figure 8. Traveling waves in two translating subsystems coupled to a stationary constraint.

are shown in Figure 7. The energy dissipated by the damping element is shown to be maximized at $d_{opt} = 2$ and its magnitude is one half of the incident energy.

5. FREE VIBRATION ANALYSIS BY TRAVELING WAVES

The natural modes of vibration (standing waves) of any continuous system consist of the superposition of equal but opposite traveling waves. In this part, the natural frequencies of a translating string are determined by the traveling wave method.

5.1. NATURAL FREQUENCIES OF A TRANSLATING STRING

In a translating string, the total phase change is γ_d , as a wave propagates from the upstream boundary to the downstream one. The phase of an upstream wave changes by γ_u over the length of the string. The total phase change, as the wave travels downstream and upstream, becomes

$$\gamma_d + \gamma_u + \phi_d + \phi_u, \quad (44)$$

where ϕ_d and ϕ_u are phase differences at the downstream and upstream ends. From the phase-closure principle [26, 27], if the total phase change is an integer multiple of 2π , the condition describes a natural frequency of the system. For an unconstrained translating string with fixed supports, the total phase differences satisfying the principle,

$$\omega \left(\frac{1}{1+v} + \frac{1}{1-v} \right) + \pi + \pi = 2\pi n, \quad (45)$$

give the natural frequencies of the classical moving threadline,

$$\omega_n = n\pi(1 - v^2), \quad (46)$$

where $n = 1, 2, 3, \dots$.

5.2. NATURAL FREQUENCIES IN A CONSTRAINED TRANSLATING STRING

If a constraint is coupled to the translating string at $x = x_c$, the string-constraint system is modelled by two coupled strings with lengths x_c and $1 - x_c$, as shown in Figure 8. The natural frequency of the constrained string is determined by traveling waves in keeping with reference [28] for two coupled non-translating subsystems. The free vibration of a particular natural frequency ω is expressed in terms of four traveling waves as

$$w_A(x, t) = A_d e^{i(\omega t - \gamma_d x)} + A_u e^{i(\omega t + \gamma_u x)}, \quad w_B(x, t) = B_d e^{i(\omega t - \gamma_d x)} + B_u e^{i(\omega t + \gamma_u x)}. \quad (47)$$

These wave equations must satisfy appropriate scattering conditions to represent a natural frequency of the two coupled strings. At the coupling point $x = 0$, the following conditions are satisfied:

$$B_d = tA_d + rB_u, \quad A_u = tB_u + rA_d, \quad (48)$$

where t and r are the transmission and reflection coefficients at the constraint. Without any external source of energy in the constraint, the coefficients satisfy equations (11) and (12). The reflection conditions at the downstream and upstream fixed boundaries are

$$r_d = \frac{B_u e^{i\gamma_u(1-x_c)}}{B_d e^{-i\gamma_d(1-x_c)}} = -1, \quad r_u = \frac{A_d e^{i\gamma_d x_c}}{A_u e^{-i\gamma_u x_c}} = -1. \quad (49)$$

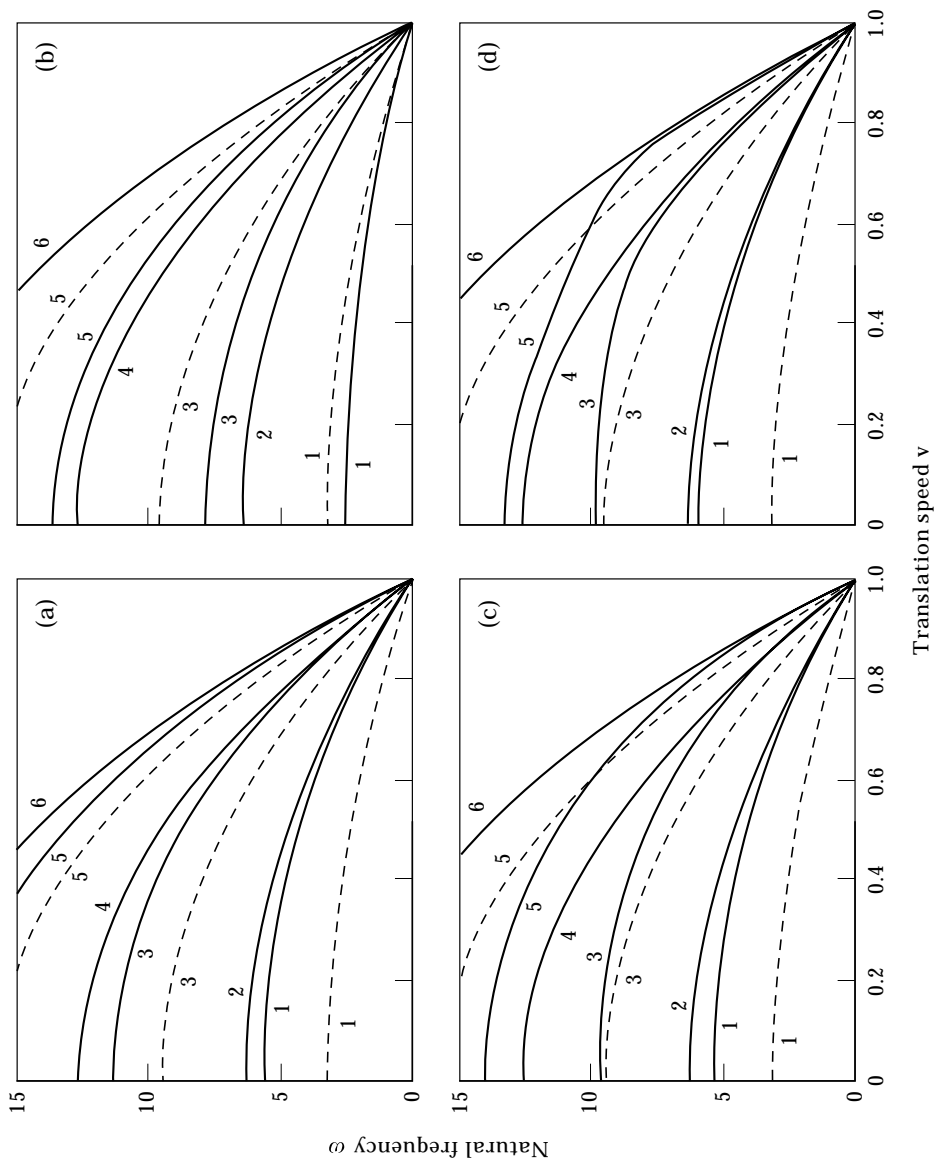


Figure 9. Natural frequency loci of the first six modes of a translating string coupled to a spring-mass oscillator at $x = 0.5$ (solid lines). Dashed lines are the solution of an unconstrained case ($m = k = 0$): (a) $m = 0, k = 30$; (b) $m = 0.3, k = 0$; (c) $m = 0.3, k = 30$; (d) $m = 1, k = 100$.

Equations (48) and (49) yield

$$\begin{pmatrix} 1 - r\mu_d & -t\mu_u \\ -t\mu_d & 1 - r\mu_u \end{pmatrix} \begin{pmatrix} B_d \\ A_u \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (50)$$

where $\mu_u = r_u e^{-ix_c(\gamma_d + \gamma_u)} = e^{i(\pi - x_c(\gamma_d + \gamma_u))}$ and $\mu_d = r_d e^{-i(1-x_c)(\gamma_d + \gamma_u)} = e^{i(\pi - (1-x_c)(\gamma_d + \gamma_u))}$. The non-trivial solution for B_d and A_u requires

$$(1 - r\mu_d)(1 - r\mu_u) - t^2\mu_u\mu_d = 0. \quad (51)$$

For the case of the non-dissipative constraint ($d = 0$), equation (51) is expressed in the simpler form

$$\cos \frac{\psi_u + \psi_d}{2} = |r| \cos \frac{\psi_u - \psi_d}{2}, \quad (52)$$

where

$$\psi_u = \pi + \phi_r - x_c(\gamma_d + \gamma_u), \quad \psi_d = \pi + \phi_r - (1 - x_c)(\gamma_d + \gamma_u). \quad (53)$$

With $\gamma_d + \gamma_u = 2\omega/(1 - v^2)$, equation (52) gives the natural frequencies of the string coupled to the undamped constraint.

Using equation (52), eigenfrequency loci are plotted in Figure 9, as v varies from zero to the critical speed. Here the spring-mass constraint is located at $x = 0.5$ and the

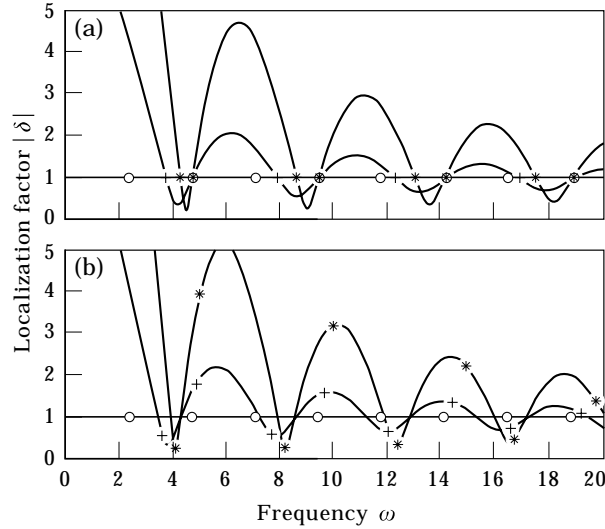


Figure 10. Dependence of localization $|\delta|$ on frequency: (a) a spring constraint is located at $x = 0.5$; (b) at $x = 0.55$. Natural frequencies of the first eight modes are marked: o, $k = 0$; +, $k = 3$; *, $k = 30$.

magnitude and phase angle of the reflection coefficient at the constraint are

$$|r| = \frac{k - m\omega^2}{\{(k - m\omega^2)^2 + 4\omega^2\}^{1/2}}, \quad \phi_r = -\arctan\left(\frac{2\omega}{k - m\omega^2}\right). \quad (54)$$

The dashed lines represent the frequencies of an unconstrained string ($k = m = 0$). The natural frequencies of the even numbered modes are unchanged for all the cases, because they vanish at the coupling point. The natural frequencies of the odd numbered modes increase with spring stiffness (Figure 9(a)) and decrease with mass (Figure 9(b)). For two cases of a spring-mass constraint satisfying $\omega_c = \sqrt{k/m} = 10$, natural frequency loci are shown in Figures 9(c) and (d). The odd numbered frequencies increase below ω_c , and decrease above ω_c . The frequency loci always cross those of an unconstrained string at $\omega = \omega_c$. By an eigenvalue inclusion principle for distributed gyroscopic systems [29], the natural frequencies ω_n of the string coupled to the mass-spring constraint at any point ($0 \leq x_c \leq 1$) satisfy

$$\omega_r^n \leq \omega_n \leq \omega_{n+1}^u \text{ if } \omega_n < \omega_c, \quad \omega_n \leq \omega_n^u \leq \omega_{n+1} \text{ if } \omega_n > \omega_c, \quad (55)$$

where $n = 1, 2, 3, \dots$ and ω_n^u are natural frequencies of an unconstrained string. The variation in the natural frequencies caused by adding a constraint is also studied in the literature [10].

6. MODE LOCALIZATION

6.1. LOCALIZATION FACTOR

Structural irregularities in a symmetric continuum, depending on the magnitude of disorder and the strength of coupling, may localize the vibration modes and confine vibrating energy to a certain region. In a translating string subjected to a stationary constraint, the constraint location x_c may cause asymmetry (disorder) of the system. The vibration localized into downstream and upstream segments of the translating string is quantified by the *localization factor* [28]. From equations (50) and (51), the ratio of vibrations in the two subsystems is expressed by

$$\delta = \sqrt{\frac{B_d B_u}{A_d A_u}} = \sqrt{\frac{B_d^2 \mu_d}{A_u^2 \mu_u}} = \frac{1 - r\mu_u}{t\sqrt{\mu_d \mu_u}}. \quad (56)$$

This ratio is real when evaluated at a natural frequency. Using $r\mu_u = |r| e^{i\psi_u}$ and $|t|^2 = 1 - |r|^2$, the ratio (56) takes the simpler form

$$|\delta| = \frac{|1 - r\mu_u|}{|t|} = \sqrt{\frac{1 + |r|^2 - 2|r| \cos \psi_u}{1 - |r|^2}}. \quad (57)$$

The vibration ratio $|\delta|$ gives the *degree of localization* of the two subsystems coupled by the undamped constraint ($d = 0$). When evaluated at a particular natural frequency, it determines the degree of localization of the corresponding natural mode. When $|r| = 0$, the localization factor is always unity and the vibration modes are equally distributed.

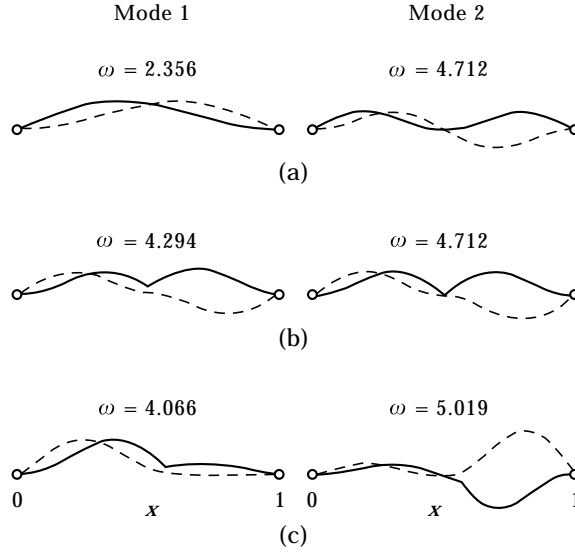


Figure 11. Localization of the first two vibration modes by a spring constraint: (a) unconstrained case, $k = 0$; (b) spring with $k = 30$ located at $x_c = 0.5$; (c) spring with $k = 30$ located at $x_c = 0.55$. The solid and dashed curves are real and imaginary parts of the modes, respectively.

6.2. BOUND AND SENSITIVITY OF MODE LOCALIZATION

From equation (57), the localization factor δ has the bounds

$$\sqrt{\frac{1-|r|}{1+|r|}} \leq |\delta| \leq \sqrt{\frac{1+|r|}{1-|r|}}. \quad (58)$$

It is noted that the upper and lower bounds of the degree of localization are determined in terms of the reflection coefficient r at the coupling point. The sensitivity of the localization factor to the constraint location is also obtained as

$$S = \frac{d|\delta|}{dx_c} = \frac{|r| \sin \psi_u (\gamma_d + \gamma_u)}{(1 + |r|^2 - 2|r| \cos \psi_u)^{1/2} (1 - |r|^2)^{1/2}}. \quad (59)$$

The sensitivity S of the mode localization actually quantifies the strength of *curve veering* of eigenfrequency loci.

6.3. MODE LOCALIZATION BY SPRING CONSTRAINT

With the magnitude and phase angle of the reflection coefficient at a spring constraint,

$$|r| = \frac{k}{(k^2 + 4\omega^2)^{1/2}}, \quad \phi_r = -\arctan \frac{2\omega}{k}, \quad (60)$$

the localization factor $|\delta|$ is plotted as a function of frequency when $k = 0, 10$ and 30 in Figure 10. The first six natural frequencies are calculated from equation (52) and marked (o, + and *). The amplitudes of the oscillatory curves decrease with frequency. The effect of the constraint on the mode localization is strong at low frequency and large spring stiffness. When the elastic constraint is located at $x = 0.5$ (Figure 10(a)), natural frequencies of the even modes remain unaltered. The odd numbered frequencies increase and approach the even numbered ones as k increases. As $k \rightarrow \infty$, the system develops repeated natural frequencies at the even numbered frequencies. In all the cases,

the localization factor $|\delta|$ remains unity because the constraint is symmetric. If the string has a structural irregularity by locating the constraint at $x_c = 0.55$, the system shows localization of natural modes. Figure 10(b) shows that the odd numbered modes are localized in the upstream subsystem, and the even modes are confined in the downstream one. The effect of the disorder on the mode shapes increases with k and decreases with frequency. The first and second modes of free response are shown in Figure 11 for three different cases. Both real and imaginary parts of the complex eigenfunctions are localized for the case of $x_c = 0.55$ (Figure 11(c)). Small changes in the constraint location may produce large changes in the modal response of the translating string.

The degrees of localization of the first three modes and the frequency spectrum as a function of constraint location x_c are plotted in Figures 12(a) and (b). Here, spring stiffness is $k = 30$ and $v = 0.5$. In Figure 12(b), dashed lines are at the limiting case $k \rightarrow \infty$ with natural frequencies

$$\omega_m = \frac{m\pi(1-v^2)}{x_c}, \quad \omega_n = \frac{n\pi(1-v^2)}{1-x_c}, \quad (61)$$

where $m, n = 1, 2, 3, \dots$. Veering of two frequency loci occurs around the regions where frequency loci (dashed lines) of the decoupled system cross (i.e., repeated frequencies). Further, the localization factors of the vibration modes cross the line of $|\delta| = 1$ at the curve veering regions. An important characteristic of curve veering is that the eigenfunctions are interchanged during veering in a rapid but continuous way [20]. The vibration modes associated with curve veering are highly localized in either the downstream or upstream segment. The strength of curve veering is quantified by the sensitivity S in equation (59) in terms of the magnitude of disorder x_c and the coupling factor k . The degree of localization weakens for higher frequency modes. Both the reflection coefficient r and the upper bound of $|\delta|$ in equation (58) decrease with increasing frequency. The free response

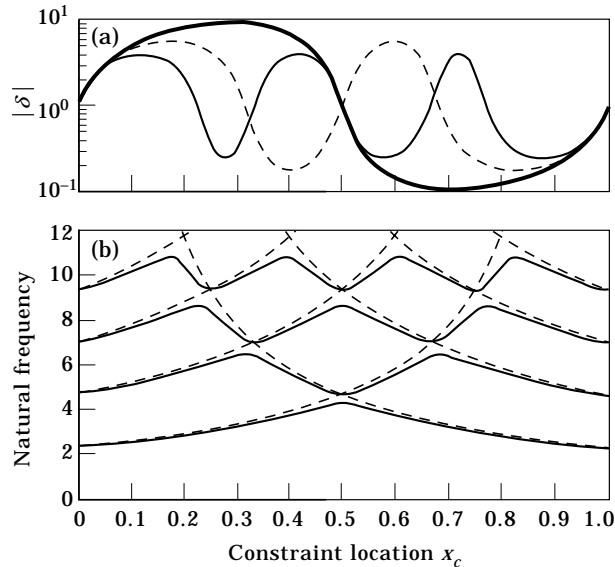


Figure 12. Mode localization and natural frequency loci of an elastically constrained string as a function of the constraint location when $k = 30$ and $v = 0.5$: (a) localization factor $|\delta|$ of the first three modes: —, mode 1; ---, mode 2; —·—, mode 3; (b) frequency loci: —, $k=30$; ---, $k \rightarrow \infty$.

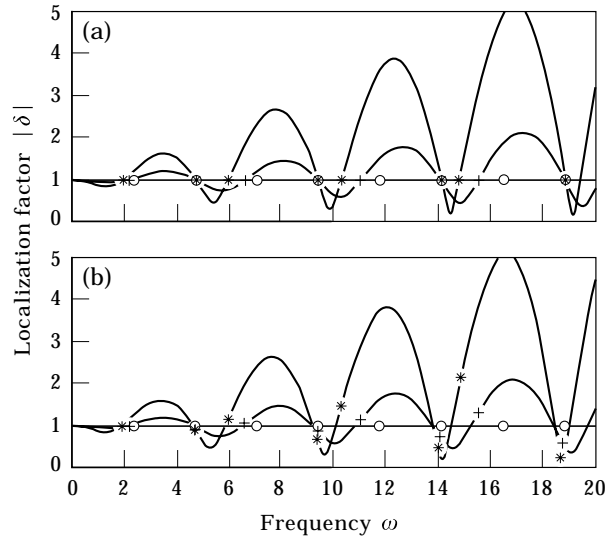


Figure 13. Dependence of localization factor $|\delta|$ on frequency: (a) an inertia constraint located at $x = 0.5$; (b) at $x = 0.51$. Natural frequencies of the first eight modes are marked: o, $m = 0$; +, $m = 0.1$; *, $m = 0.3$.

of the system is sensitive to small irregularity in constraint location around the regions with curve veering.

6.4. MODE LOCALIZATION BY MASS CONSTRAINT

The localization factor $|\delta|$ is plotted for the cases of $m = 0, 0.1$ and 0.3 in Figure 13. The first six vibration modes are marked (o, + and *). When the mass constraint is located at $x = 0.5$ (Figure 13(a)), the natural frequencies of the even modes remained unaltered. The odd numbered frequencies decrease and approach the even numbered ones,

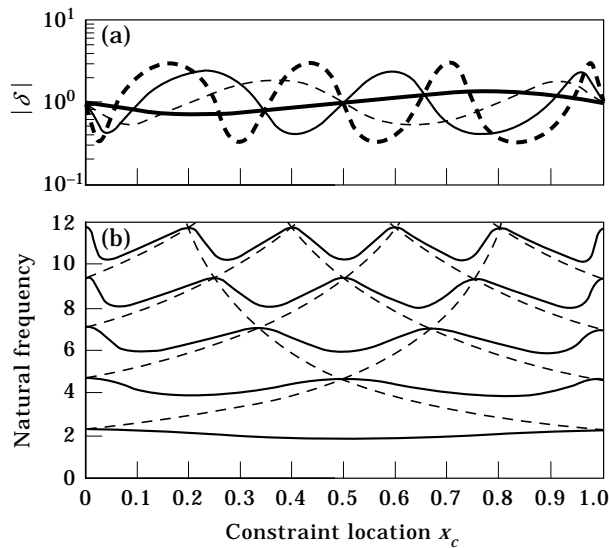


Figure 14. Mode localization and frequency loci by location of a point mass constraint of $m = 0.1$ and $v = 0.5$: (a) localization factor $|\delta|$ of the first four modes: —, mode 1; ---, mode 2; —·—, mode 3; - - - -, mode 4; (b) frequency loci: —, $m = 0.1$; ---, $m \rightarrow \infty$.

as m increases, and the degree of localization of each mode is unity. With braking of symmetry ($x_c = 0.51$), the vibration modes are localized in either downstream or upstream segments and the localization factor increases with frequency and the constraint mass. The localization factors and frequency loci of the first four modes are plotted in Figures 14(a) and (b). Unlike the case of a spring constraint, the mode localization and veering of frequency loci by the mass constraint are strong for high frequency modes. From equation (36), high frequency modes ($r \rightarrow -1$ and $t \rightarrow 0$) are localized strongly for small structural irregularities caused by the constraint location. $|\delta|$ becomes unbounded as ω goes to infinity ($0 \leq |\delta| \leq \infty$).

6.5. MODE LOCALIZATION BY DAMPED CONSTRAINT

When a translating string is in contact with damping, the coupled string has complex eigenvalues ($\lambda = v + i\omega$). The derivative of the n th eigenvalue λ_n with respect to the damping coefficient [10],

$$\frac{\partial \lambda_n}{\partial d} = -(1 - v^2) \sin^2(n\pi x_c), \quad (62)$$

is negative real. Therefore, free vibration is stabilized by the damped constraint. The complex natural frequencies and mode localization of the damped string are not obtained from equations (52) and (57), because the equations are valid for the undamped case ($d = 0$). The effect of the damped constraint on the mode localization of the constrained string is not reported in the literature. The interesting topic will be further studied by the authors using the traveling wave technique described in this paper.

7. CONCLUSIONS

The energetics and mode localization of free vibration in a translating string coupled to a constraint of a spring-mass-damper system are studied. The main findings of this paper are summarized as follows.

First, the string tension and the non-conservative centrifugal force at the constraint work on the traveling string. The resultant energy flux at the coupling point is quantified by the energy flux coefficient. Energy transferred into the string over a cycle is always positive for a forward wave and is always negative for a backward one.

Second, a traveling wave of high frequency transmits through an elastic constraint and one of low frequency reflects from the constraint. The wave transmission and reflection at a point mass constraint are opposite to the case of the elastic constraint. For a spring-mass constraint, the incident wave completely transmits when the natural frequency of the oscillator ($\sqrt{k/m}$) equals the incident wave frequency.

Third, wave scattering characteristics at a damped constraint is independent of wave frequency. The energy dissipated by damping is quantified by the absorption coefficient and the maximum energy dissipation is 50% of the incoming wave energy. The corresponding damping coefficient is twice the impedance of the string; $d = 2\mathcal{Z}$.

Finally, structural asymmetry, produced by attaching the constraint, localizes vibration modes of the translating string. The degree of vibration localized into a downstream or upstream segment depends on the reflection coefficient and location of the constraint. Under a spring constraint, low frequency modes are strongly localized. The degree of localization by a point mass constraint increases with increasing frequency.

REFERENCES

1. J. A. WICKERT and C. D. MOTE, JR. 1988 *Shock and Vibration Digest* **20**, 3–13. Current research on the vibration and stability of axially moving materials.
2. K. W. WANG and S. P. LIU 1991 *Shock and Vibration Digest* **23**, 8–13. On the noise and vibration of chain drive system.
3. M. P. PAIDOUSSIS and G. X. LI 1993 *Journal of Fluids and Structures* **7**, 137–204. Pipes conveying fluid: a model dynamical problem.
4. N. C. PERKINS 1990 *American Society of Mechanical Engineers, Journal of Vibration and Acoustics* **112**, 2–7. Linear dynamics of a translating string on an elastic foundation.
5. C. A. TAN and L. ZHANG 1994 *American Society of Mechanical Engineers, Journal of Vibration and Acoustics* **116**, 318–325. Dynamic characteristics of a constrained string translating across an elastic foundation.
6. S.-P. CHENG and N. C. PERKINS 1991 *Journal of Sound and Vibration* **144**, 281–292. The vibration and stability of a friction-guided, translating string.
7. J. A. WICKERT and C. D. MOTE, JR. 1991 *Journal of Sound and Vibration* **149**, 267–284. Traveling load response of an axially-moving string.
8. W. ZHU and C. D. MOTE, JR. 1993 *Journal of Sound and Vibration* **177**, 591–610. Free and forced response of an axially moving string transporting a damped linear oscillator.
9. W. ZHU and C. D. MOTE, JR. 1995 *American Society of Mechanical Engineers, Journal of Applied Mechanics* **62**, 873–879. Propagation of boundary disturbances in an axially moving strip in contact with rigid and flexible constraints.
10. J.-S. CHEN 1997 *American Society of Mechanical Engineers, Journal of Vibration and Acoustics* **119**, 152–157. Natural frequencies and stability of an axially-traveling string in contact with a stationary load system.
11. G. S. SCHAJER 1984 *Journal of Sound and Vibration* **92**, 11–19. The vibration of a rotating circular string subject to a fixed elastic restraint.
12. Y. XIONG and S. G. HUTTON 1994 *Journal of Sound and Vibration* **169**, 669–683. Vibration and stability analysis of a multi-guided rotating string.
13. L. YANG and S. G. HUTTON 1995 *Journal of Sound and Vibration* **185**, 139–154. Interactions between an idealized rotating string and stationary constraints.
14. T. CHUBACHI 1958 *Bulletin Japan Society of Mechanical Engineers* **1**, 24–29. Lateral vibration of axially moving wire or belt form materials.
15. W. L. MIRANKER 1960 *IBM Journal Research and Development* **4**, 36–42. The wave equation in a medium in motion.
16. J. A. WICKERT and C. D. MOTE, JR. 1989 *Journal of the Acoustical Society of America* **85**, 1365–1368. On the energetics of axially moving continua.
17. S.-Y. LEE and C. D. MOTE, JR. 1997 *Journal of Sound and Vibration* **204**, 735–753. A generalized treatment of the energetics of translating continua, part I: strings and second-order tensioned pipes.
18. C. H. HODGES and J. WOODHOUSE 1983 *Journal of the Acoustical Society of America* **74**, 894–905. Vibration isolation from irregularity in a nearly periodic structure: theory and measurements.
19. C. PIERRE and E. H. DOWELL 1987 *Journal of Sound and Vibration* **114**, 549–564. Localization of vibrations by structural irregularity.
20. N. C. PERKINS and C. D. MOTE, JR. 1986 *Journal of Sound and Vibration* **106**, 451–463. Comments on curve veering in eigenvalue problems.
21. F. R. ARCHIBALD and A. G. EMSLIE 1958 *American Society of Mechanical Engineers, Journal of Applied Mechanics* **25**, 347–348. The vibration of a string having a uniform motion along its length.
22. R. D. SWOPE and W. F. AMES 1963 *Journal of the Franklin Institute* **275**, 36–55. Vibrations of a moving threadline.
23. B. R. MACE 1992 *Journal of Sound and Vibration* **154**, 289–319. Power flow between two continuous one-dimensional subsystems: a wave solution.
24. R. W. GREGORY and M. P. PAIDOUSSIS 1966 *Proceedings of the Royal Society of London* **293A**, 512–544. Unstable oscillation of tabular cantilevers conveying fluid, I. Theory and II. Experiment.
25. S.-Y. LEE and C. D. MOTE, JR. 1996 *American Society of Mechanical Engineers, Journal of Dynamic Systems, Measurement and Control* **118**, 66–74. Vibration control of an axially moving string by boundary control.
26. L. CREMER, M. HECKEL and E. E. UNGAR 1988 *Structure-Borne Sound*. Berlin: Springer-Verlag.

27. D. J. MEAD 1994 *Journal of Sound and Vibration* **171**, 695–702. Waves and modes in finite beams: application of the phase-closure principle.
28. R. S. LANGLEY 1995 *Journal of Sound and Vibration* **185**, 79–91. Mode localization up to high frequencies in coupled one-dimensional subsystems.
29. B. YANG 1992 *American Society of Mechanical Engineers, Journal of Applied Mechanics* **59**, 650–656. Eigenvalue inclusion principles for distributed gyroscopic systems.